Dynamics of magnetic Bianchi VI_0 cosmologies

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Abstract

Methods of dynamical systems analysis are used to show rigorously that the presence of a magnetic field orthogonal to the two commuting Killing vector fields in any spatially homogeneous Bianchi type VI_0 vacuum solution to Einstein's equation changes the evolution toward the singularity from convergent to oscillatory. In particular, it is shown that the α-limit set (for time direction that puts the singularity in the past) of any of these magnetic solutions contains at least two sequential Kasner points of the BKL sequence and the orbit of the transition solution between them. One of the Kasner points in the α-limit set is non-flat, which leads to the result that each of these magnetic solutions has a curvature singularity.

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1 Introduction

Progress in the characterization of the asymptotic behavior of spatially homogeneous solutions to Einstein's equation has been made by the use of methods from dynamical systems analysis. (See [1] for an overview.) For example, these methods have been used to show that solutions in various spatially homogeneous subclasses are convergent or oscillatory in a given time direction. The vacuum Bianchi type VI_0 spacetimes are a class of spatially homogeneous solutions which are known to be convergent in both time directions [2]. One time direction is singular and the other is not. In these spacetimes the three linearly independent Killing vector fields can be chosen so that two of them commute with each other. In [3] evidence is presented that provides strong support for the conjecture that if there is a magnetic field present which is orthogonal to the two commuting Killing vector fields then the approach to the singularity is no longer convergent but instead oscillatory, and moreover, generically mixmaster. This is corroborated by the numerical study reported in [4].

Rigorous results concerning mixmaster dynamics have been elusive. Classes of spatially homogeneous solutions believed to be mixmaster are Bianchi type IX, Bianchi type VIII and magnetic Bianchi type VI_0, with or without a perfect fluid in each case. (See [5, 6] for additional mixmaster classes.) Not only has the occurrence of mixmaster dynamics remained a conjecture, but it has not even been shown that solutions in the classes thought to be mixmaster are oscillatory. Progress toward such a result for vacuum Bianchi type IX and vacuum Bianchi type VIII was made in [2]. The magnetic Bianchi type VI_0 spacetimes considered in [3] are considered again in the present paper (without a perfect fluid) and, building on the results of [3] by using techniques developed in [2], it is rigorously shown that all of them are oscillatory in the singular time direction.

The methods used to show this are those of dynamical systems analysis. A discussion of these methods can be found in [1] and in references cited therein. In particular, the formulation of the magnetic Bianchi VI_0 spacetimes as a dynamical system is taken from [3], and definitions and results from [3] and [2] are used throughout. Three types (convergent, oscillatory, mixmaster) of asymptotic behavior are discussed in section 2. A review of results from [3] is given in section 3. The proof of the

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new result that magnetic Bianchi VI_0 solutions are oscillatory is presented in section 4. A method which might lead to a proof of mixmaster behavior in this class of spacetimes is suggested in section 5.

Knowledge of the asymptotic behavior of a solution can lead to an answer to the question of whether or not an extension through the initial singularity is possible. In section 6 it is shown that each of these solutions has a curvature singularity. Therefore no extension is possible. In the course of the analysis, several other types of spatially homogeneous solutions to Einstein's equation are considered. Which of these have curvature singularities is also determined in section 6. The solutions considered here which can be extended past the initial singularity, and therefore do not have a curvature singularity, include rotationally symmetric solutions of Bianchi type I with a magnetic field orthogonal to the plane of rotational symmetry. An extension of these spacetimes is given in the appendix. Section 7 consists of concluding remarks.

2 Types of asymptotic behavior

In a spatially homogeneous solution to Einstein's equation there is a foliation by homogeneous spatial hypersurfaces with time coordinate t constant on each hypersurface. Assume that the foliation of homogeneous spatial hypersurfaces is maximally extended. Choose t so that t takes on all real values. Define a generalized Kasner exponent, $p_i(t)$, to be an eigenvalue of the extrinsic curvature of the homogeneous hypersurface at time t divided by the mean curvature of the homogeneous hypersurface [7]. In a Kasner (vacuum Bianchi type I) spacetime the generalized Kasner exponents are exactly the Kasner exponents and they are constant in time. There are other spatially homogeneous solutions to Einstein's equation, for example the spatially flat Friedmann-Robertson-Walker models, whose generalized Kasner exponents are constant in time, but do not satisfy the Kasner relation, $p_1^2 + p_2^2 + p_3^2 = 1$. A convergent solution (in a particular time direction) is one in which each of the three generalized Kasner exponents converges to a limit in that time direction [7]. For example, if a solution is convergent in the negative time direction, then $\lim_{t\to-\infty} p_i(t)$ exists. A solution may be convergent in both time directions, in only one time direction, or in neither.

When a class of spatially homogeneous solutions is studied as a dynamical system using the expansion normalized variables of Wainwright and collaborators [1] then a convergent solution is one whose α-limit set (for the negative time direction, or ω-limit set for the positive time direction) consists of a single equilibrium point of the system. Such an equilibrium point may be a Kasner point, which represents a Kasner solution, or it may represent another solution whose generalized Kasner exponents are constant so that, as in the Kasner solutions, the evolution is completely characterized by the volume expansion. An effect of the normalization by the volume expansion is that each equilibrium point corresponds to an entire spacetime and each one-dimensional orbit corresponds to a one-parameter family of conformally related spacetimes [8]. Since the volume expansion is zero at the moment of maximum volume in spacetimes of Bianchi type IX, the expansion normalized variables represent half of the evolution – either the expanding phase or the collapsing phase, and so the limit set of a solution in the direction of maximum expansion gives information about the evolution at the moment of maximum expansion. A magnetic Bianchi type VI_0 spacetime is either always expanding or always collapsing, depending on choice of time direction, so the expansion normalized variables cover the entire spatially homogeneous evolution.

An oscillatory solution is one whose α-limit set (or ω-limit set) contains at least two equilibrium points of the system. In all known cases, the equilibrium points contained in such a limit set are a finite number of isolated Kasner points and the limit set is a heteroclinic cycle. A heteroclinic cycle is a set of n distinct equilibrium points, q_i, and n orbits of the dynamical system, Γ_i, such that each orbit is convergent in both time directions and the ω-limit set of Γ_i consists of the point q_i while if $i < n$ the α-limit set of Γ_i consists of the point q_{i+1} and if $i = n$ the α-limit set of Γ_i consists of the point q_1.

Qualitative analysis and the results of numerical simulations give the following picture for the evolution toward the singularity in generic solutions belonging to the classes of spacetimes believed

to be mixmaster. Let S be a nonexceptional solution in one of these classes. A sequence of Kasner evolutions, q_i, approximates the evolution of S. Given q_i, then q_{i+1} is determined by the BKL map [9, 10, 11, 12]. Moreover, when the evolution of S is not approximately Kasner, during the transition from q_i to q_{i+1}, the evolution of S is approximated by a spatially homogeneous solution to Einstein's equation which is uniquely determined by q_i. In the case of Bianchi types VIII and IX this is a vacuum Bianchi type II solution. In the case of magnetic Bianchi type VI_0 some of the transitions are approximated by vacuum solutions of Bianchi type II and some are approximated by magnetic solutions of Bianchi type I [3]. The solution approximating the evolution during the transition from one Kasner to the next converges to q_i to the future and to q_{i+1} to the past (for time direction that puts the singularity in the past). As the singularity is approached the approximation of the evolution of S by such a sequence of Kasners along with the transition solutions determined by the sequence improves. However, it is not necessarily expected that the evolution of S converges to a single BKL sequence of Kasners along with the transition solutions determined by the sequence [2]. It may be that the evolution always eventually diverges from one sequence and another one becomes a better approximation.

The description of mixmaster evolution just given can be stated in terms of the α-limit set of S. The presence of a perfect fluid does not significantly alter the picture. If there is no perfect fluid, then each mixmaster class is a four-dimensional invariant set of a dynamical system and the solution, S, is represented by an orbit in the four-dimensional invariant set. The boundary of this four-dimensional set is in each case the union of the three-dimensional sets listed below and the compact two-dimensional invariant set, $\mathcal{A}$, consisting of the Kasner points along with the orbits of all transition solutions between sequential Kasner points of the BKL sequence. The boundary of vacuum Bianchi type IX consists of orbits of vacuum Bianchi types VII_0 (a three-dimensional set), II (a two-dimensional set) and I (the Kasners). In this case $\mathcal{A}$ is the union of the latter two sets. The boundary of vacuum Bianchi type VIII consists of orbits of vacuum Bianchi types VII_0, VI_0 (a three-dimensional set), II and I. $\mathcal{A}$ is again the union of the latter two sets. The boundary of magnetic Bianchi type VI_0 consists of orbits of vacuum Bianchi type VI_0, magnetic solutions of Bianchi types II (a three-dimensional set) and I (a two-dimensional set) and vacuum Bianchi types II and I. In this case $\mathcal{A}$ is the union of the latter three sets. It is known that the α-limit set of S must be contained in the boundary of the four-dimensional invariant set in which it lies [3, 8]. If S is magnetic Bianchi type VI_0 then it is known that its α-limit set is non-empty since the orbit is contained in a compact set of the dynamical system. If S is Bianchi type VIII or IX then until recently it has not been known that its α-limit set is non-empty. (It appears that this is no longer the case, due to a recent result by Ringström showing that such an α-limit set is indeed non-empty [13].) To be consistent with the picture of mixmaster dynamics, the α-limit set of S should be non-empty, and it should be contained in just part of the boundary, obtained by excluding the three-dimensional sets. That is, it should be contained in the set $\mathcal{A}$. This is because in order for the approximation of the evolution of S by a BKL sequence of Kasners along with the transition solutions determined by the Kasners in the sequence to get better as the singularity is approached, the orbit of S must eventually remain within any compact set containing a neighborhood of $\mathcal{A}$. Given any point, p, of the boundary that is not in $\mathcal{A}$, such a compact set can be found which does not contain p. Furthermore, if it could be shown that the α-limit set of S is contained in the set $\mathcal{A}$, then the results in [2] and in the present paper would show that an entire BKL sequence of Kasner points along with the transition solutions between them must lie in the α-limit set of S. Whether $\mathcal{A}$ or a subset of $\mathcal{A}$ is the α-limit set of generic solutions in the mixmaster classes would still be an open question.

3 Magnetic Bianchi VI_0

The union of the orbits of magnetic Bianchi VI_0 solutions is the four-dimensional open invariant set, U, defined by [3]:

$$\Sigma_+^2 + \Sigma_-^2 + N_-^2 + \frac{3}{2}H^2 = 1, \tag{1}$$

$$N_- > 0, \quad N_+^2 < 3N_-^2, \quad \text{and} \quad H > 0. \tag{2}$$

This formulation is in terms of the shear tensor, the structure constants and the magnetic field in an orthonormal frame which is invariant under the action of the group of isometries. H is obtained from the magnetic field, which is orthogonal to the two commuting killing fields. Σ_+ and Σ_- are obtained from the shear tensor while N_+ and N_- are obtained from the structure constants. These quantities have been normalized by the expansion scalar. Equation 1 is the Hamiltonian constraint. It is equation (2.19) in [3] with $\Omega = 0$. See [3] for further details concerning this construction and its representation of all solutions in this class.

The time direction is chosen so that the singularity is in the past (τ decreasing). The evolution equations are:

$$\begin{aligned}
\Sigma_+' &= -2N_-^2(1+\Sigma_+) + \frac{3}{2}H^2(2-\Sigma_+), \\
\Sigma_-' &= -(2N_-^2 + \frac{3}{2}H^2)\Sigma_- - 2N_+N_-, \\
N_+' &= (2\Sigma_+(1+\Sigma_+) + 2\Sigma_-^2 + \frac{3}{2}H^2)N_+ + 6\Sigma_-N_-, \\
N_-' &= (2\Sigma_+(1+\Sigma_+) + 2\Sigma_-^2 + \frac{3}{2}H^2)N_- + 2\Sigma_-N_+, \\
H' &= -(\Sigma_+(2-\Sigma_+) - \Sigma_-^2 + N_-^2)H.
\end{aligned} \tag{3}$$

These equations can be obtained from equations (2.17) in [3] by using their equation (2.16) with $\Omega = 0$. Evolution satisfying equations 3 preserves the Hamiltonian constraint.

There is exactly one equilibrium point in U, the point at which $\Sigma_+ = -\frac{1}{4}$, $\Sigma_- = 0$, $N_+ = 0$, $N_- = \frac{3}{4}$ and $H = \frac{1}{2}$. In [3] it is shown that the ω-limit set of each solution in U consists of this single equilibrium point. That is, all of these magnetic Bianchi VI_0 solutions are convergent in the non-singular time direction.

The boundary of U, ∂U, is made up of various sets which are invariant with respect to equations 3. Solutions to equations 3 in these invariant sets represent solutions to Einstein's equation in various other spatially homogeneous classes. As mentioned previously, these are vacuum Bianchi VI_0 ($H = 0$), magnetic Bianchi II ($N_+^2 = 3N_-^2$) and I ($N_- = 0$), vacuum Bianchi II ($H = 0$ and $N_+^2 = 3N_-^2$) and the Kasner points ($H = 0$ and $N_- = 0$). Note that the magnetic field is not the most general one possible. See [5, 6] for other possibilities. Each Kasner point is an equilibrium point and there are no other equilibrium points in ∂U. Three of the Kasner points represent spacetimes which are isometric to a part of Minkowski space, and so are called the flat Kasner spacetimes. These are labeled T_i, $i \in 1,2,3$ (see figure 1).

The orbit structure in the two-dimensional invariant sets of the boundary, vacuum Bianchi II and magnetic Bianchi I, has been known for some time [3, and references cited therein]. Each of these solutions is convergent in both time directions. These are the transition solutions from one Kasner point to the next in the BKL map. There are two copies of vacuum Bianchi II in ∂U. On one copy (which will be denoted vacuum Bianchi II^+) $N_+ = \sqrt{3}N_-$ and on the other (which will be denoted vacuum Bianchi II^-) $N_+ = -\sqrt{3}N_-$. The Kasner circle is the boundary of each of these three two-dimensional open sets. The union of these three sets and the Kasner circle is the set $\mathcal{A}$ for magnetic Bianchi VI_0.

Table 1: Strictly monotonic functions

	magnetic Bianchi II^+	magnetic Bianchi II^-	vacuum Bianchi II^+	vacuum Bianchi II^-	magnetic Bianchi I
L	decreasing	increasing	decreasing	increasing	constant
W^+	increasing	indefinite	constant	decreasing	increasing
W^-	indefinite	decreasing	increasing	constant	decreasing

The orbit structure can be seen by considering the behavior of the following three functions on solutions.

$$
\begin{aligned}
L &= \frac{\Sigma_-}{2-\Sigma_+} \\
W^+ &= \frac{1+\Sigma_+}{\sqrt{3}+\Sigma_-} \\
W^- &= \frac{1+\Sigma_+}{-\sqrt{3}+\Sigma_-}
\end{aligned}
\tag{4}
$$

Calculation of the time derivative of each of these functions, using equations 3 restricted to each set in turn, shows that on each set one of these functions is constant on solutions, one is strictly increasing on solutions and one is strictly decreasing on solutions (see table 1). Thus the orbits are straight lines when projected into the Σ_+–Σ_- plane, and it can be determined which Kasner point is in the α-limit set and which Kasner point is in the ω-limit set of each solution. The ω-limit set of a solution in magnetic Bianchi I consists of a point of $\mathcal{K}_1$, and the α-limit set consists of a point of $\mathcal{K}_2 \cup T_1 \cup \mathcal{K}_3$. See figure 1 for the definition of $\mathcal{K}_i$. The ω-limit set of a solution in vacuum Bianchi II^+ consists of a point of $\mathcal{K}_2$ and the α-limit set consists of a point of $\mathcal{K}_3 \cup T_2 \cup \mathcal{K}_1$. The ω-limit set of a solution in vacuum Bianchi II^- consists of a point of $\mathcal{K}_3$ and the α-limit set consists of a point of $\mathcal{K}_1 \cup T_3 \cup \mathcal{K}_2$.

The polarized solutions in U, in which one of the degrees of freedom of the gravitational field is absent and the metric in a coordinate basis can be diagonalized [14], form a two-dimensional invariant subset of U, defined by $\Sigma_- = 0$ and $N_+ = 0$. In [3] it is shown that the α-limit set of each polarized solution in U is the heteroclinic cycle made up of the following orbits: the Kasner points T_1 and Q_1 (see figure 1 for definition of Q_i), the single vacuum polarized Bianchi VI_0 orbit, Γ_1, whose α-limit set consists of the point Q_1 and whose ω-limit set consists of the point T_1, and the orbit in magnetic Bianchi I, Γ_2, whose α-limit set is T_1 and whose ω-limit set is Q_1. Thus the polarized solutions in U are oscillatory toward the singularity. They are not mixmaster. Their evolution toward the singularity is not approximated by a BKL sequence. Γ_2 is a transition solution between two sequential points in the BKL sequence, but Γ_1 is not. If a BKL sequence includes the point T_i then it ends there. Thus, a polarized solution in U is an "exceptional solution" in this mixmaster class (see section 2).

In [3] it is shown that the α-limit set of a non-polarized solution in U must be contained in ∂U by considering the function

$$
Z = \frac{3\Sigma_-^2 + N_+^2}{3N_-^2 - N_+^2}
\tag{5}
$$

which is strictly decreasing on any non-polarized solution in U.

4 Magnetic Bianchi VI_0 solutions are oscillatory

That the magnetic Bianchi VI_0 solutions are oscillatory toward the singularity is shown by the following theorem.

Theorem. The α-limit set of a spatially homogeneous solution to Einstein's equation with Bianchi type VI_0 symmetry and a magnetic field orthogonal to the two commuting Killing vector fields contains at least two sequential Kasner points of the BKL map and the orbit of the transition solution connecting them.

Proof of Theorem. The proof of the theorem will be given in the rest of this section. The result has already been obtained for the polarized solutions in U [3], so it remains to obtain the result for the non-polarized solutions in U. It has also already been shown that the α-limit set of a non-polarized solution in U is contained in ∂U [3]. Therefore the first part of the proof of the theorem is an analysis of the orbit structure in ∂U. It is shown that each limit set of any solution in ∂U contains a Kasner point. From this it immediately follows that the α-limit set of a non-polarized solution in U contains a Kasner point, because a non-empty α-limit set is a union of orbits along with the limit sets of each orbit. Therefore if an α-limit set is contained in ∂U it must contain an entire orbit in ∂U and it must contain the orbit's limit sets. Then, through analysis of the orbit structure in the closure of U, $\overline{U}$, in the neighborhood of a Kasner point, it is shown that if the α-limit set of a solution in U contains a Kasner point, it necessarily contains a second Kasner point.

Proposition 1. Each solution in the boundary of U is convergent to a Kasner point in both time directions. If the solution is not Kasner, then the Kasner point in its α-limit set is not the Kasner point in its ω-limit set.

Proof of Proposition 1. This result is already known for the Kasner points, the vacuum Bianchi II solutions and the magnetic Bianchi I solutions that lie in ∂U.

Consider the three-dimensional open invariant set, I_3, made up of all the magnetic Bianchi II^+ ($N_+ = \sqrt{3}N_-$) orbits in ∂U. The function L is strictly decreasing on solutions in I_3 and W^+ is strictly increasing. The existence of a function that is strictly monotonic on solutions in an invariant set implies that the limit sets of the solutions must be contained in the boundary [3, Proposition A1]. The boundary of I_3 is made up of the union of all the Kasner points, the magnetic Bianchi I orbits and the vacuum Bianchi II^+ orbits. L is strictly decreasing on the latter solutions and W^+ is strictly increasing on magnetic Bianchi I solutions. A function that is strictly monotonic on a solution must be constant on each limit set of that solution [2], and a non-empty limit set is the union of orbits. Therefore each limit set of a solution in I_3 must lie in the Kasner circle. For a given value in the range of L or W^+ there are at most two Kasner points at which the function takes this value. A limit set must be connected. Therefore each limit set of a solution in I_3 must consist of a single Kasner point. The Kasner point in the α-limit set of a solution is not the the Kasner point in the ω-limit set, since a function that is strictly monotonic on a solution cannot have the same value on both limit sets of the solution.

A similar argument, using the functions L and W^- (see table 1), shows that a solution in the set I_2, made up of all the magnetic Bianchi II^- ($N_+ = -\sqrt{3}N_-$) orbits in ∂U, is convergent to a Kasner point in either time direction, and that the Kasner point in the α-limit set is not the the Kasner point in the ω-limit set.

Now consider the three-dimensional open invariant set, I_1, made up of all the vacuum Bianchi VI_0 orbits in ∂U. It was shown in [2] that each of these solutions is convergent to a Kasner point in either time direction. In the present context, since the boundary of I_1 is made up of the Kasner points and both sets of vacuum Bianchi II solutions, this result can be obtained with the same reasoning used for I_3 by considering the function Σ_+ which is strictly decreasing on solutions in I_1 and is also strictly decreasing on both sets of vacuum Bianchi II solutions. And again, the existence of the monotonic function implies that the Kasner point in the α-limit set is not the Kasner point in the ω-limit set. □

It has now been shown that each limit set of any solution in ∂U contains a Kasner point, from which it follows, as discussed above, that the α-limit set of a non-polarized solution in U contains a Kasner point. To show that this in turn implies that there is another Kasner point in the α-limit set, further analysis of the orbit structure in $\overline{U}$ will be useful.

Figure 1 shows the sign of the eigenvalues of the linearization of the differential equation about each Kasner point [3]. At each Kasner point the eigenvector corresponding to the eigenvalue λ_H is tangent to magnetic Bianchi I while the eigenvectors corresponding to the eigenvalues $\lambda_\pm$ are tangent to vacuum Bianchi II$^\pm$. The fourth eigenvalue, λ_K, vanishes at each Kasner point and its eigenvector is tangent to the Kasner circle. From the number of eigenvalues which vanish, the number which are positive and the number which are negative it follows that each flat Kasner point has a three-dimensional center manifold and a one-dimensional unstable manifold. Each non-flat Kasner point has a one-dimensional center manifold, a two-dimensional unstable manifold and a one-dimensional stable manifold. A center manifold is not necessarily unique, while the stable and unstable manifolds are unique. If the orbit structure on a center manifold is known, the orbit structure of the nonlinear system near a Kasner point can be determined. Let O_C be the intersection of a center manifold of the Kasner point with some neighborhood, N, of the Kasner point in $\overline{U}$. Let j, k and l be the dimensions of a center manifold, the unstable manifold and the stable manifold, respectively. Let $x = (x_1, ..., x_j)$, $y = (y_1, ..., y_k)$ and $z = (z_1, ..., z_l)$. Here $x_1, ...x_j$ are coordinates on O_C, $y \in R^k$ and $z \in R^l$. For some positive ϵ, let

$$O = \{(x_1, ..., x_j, y_1, ..., y_k, z_1, ..., z_l) : x \in O_C, 0 \le y_i < \epsilon \text{ and } 0 \le z_i < \epsilon\}.$$

Let $x(\tau)$ satisfy the nonlinear system restricted to 0_C. Let $y(\tau) = y_0\, e^\tau$ and $z(\tau) = z_0\, e^{-\tau}$. Then there exists an ϵ and an N such that the nonlinear system on N is topologically equivalent to this system on O [2, *Theorem 2.3*]. (See also [15, page 99].) Topological equivalence means that there is a homeomorphism, $H : N \to O$, which maps orbits in N onto orbits in O and preserves the time orientation on the orbits.

Let q be a non-flat Kasner point. Then the Kasner circle is a center manifold of q. Since the Kasner circle is made up of equilibrium points, then in this case $x(\tau)$ as defined above is independent of τ. The solutions on O are therefore known exactly. Then the topological equivalence of the nonlinear system on N to this explicitly known system on O implies the following. In N the Kasner circle is intersected by a foliation of three-dimensional leaves. Any orbit of the nonlinear system restricted to N lies entirely in one leaf. (However, the intersection of an orbit in U with N may have points on more than one leaf.) One of the leaves contains q, which is its own α-limit set and its own ω-limit set. In this leaf there is a one-dimensional stable manifold of q, tangent to the eigenvector corresponding to the negative eigenvalue. The ω-limit set of solutions on this manifold is q. There is a two-dimensional unstable manifold of q, tangent to the two eigenvectors corresponding to the positive eigenvalues. The α-limit set of solutions on this manifold is q. Now it will be shown that the stable and unstable manifolds lie in ∂U (and therefore not in U).

The one-dimensional stable manifold of q has already been found. If $q \in \mathcal{K}_1$ its stable manifold is an orbit of magnetic Bianchi I. If $q \in \mathcal{K}_2$ its stable manifold is an orbit of vacuum Bianchi II$^+$. If $q \in \mathcal{K}_3$ its stable manifold is an orbit of vacuum Bianchi II$^-$.

Now consider the closure of I_1, $\overline{I}_1$. In this three-dimensional subsystem the eigenvalues of the linearization of the differential equation about each Kasner point are $\lambda_\pm$ and λ_K, with the same eigenvectors as in the four-dimensional system. If $q \in \mathcal{K}_1$ and $\tilde{N}$ is a neighborhood of q in $\overline{I}_1$, then the nonlinear system restricted to $\tilde{N} \cap N$ is topologically equivalent to the system on O with l set to zero. Therefore, in $\tilde{N} \cap N$ the Kasner circle is intersected by a foliation of two-dimensional leaves, each of which is a two-dimensional unstable manifold of the Kasner point that it contains. Since $\overline{I}_1$ is an invariant manifold, this implies that the unstable manifold of q lies in $\overline{I}_1$.

Similar reasoning shows that if $q \in \mathcal{K}_2$, its unstable manifold lies in $\overline{I}_2$, while if $q \in \mathcal{K}_3$ its unstable manifold lies in $\overline{I}_3$.

Now let S be a non-polarized solution in U and let Λ be its α-limit set. It was shown above that Λ contains a Kasner point. Consider the case that Λ contains q. It has just been shown that S is not in either the stable or the unstable manifold of q. Moreover, analysis of the orbit structure in O shows that no other solutions there are convergent to $H(q)$ which shows that S is not convergent to q. But since q is in Λ, then there is a sequence of points on S and in N, $S(\tau_n)$, which converges to q,

with $\tau_n < \tau_{n-1}$. Let Γ_n be the orbit in N on which $S(\tau_n)$ lies. Note that Γ_n need not be in the leaf containing q, but the orbit structure in every leaf is the same and, in addition, the sequence of Γ_n's approaches the leaf containing q. Choose $\delta < \epsilon$ and define

$$O_\delta = \{(x, y_1, y_2, z) : x \in O_C, 0 \le y_i \le \delta \text{ and } 0 \le z \le \delta\}.$$

Then each $H(\Gamma_n)$ has two points in the boundary of O_δ in $H(N \cap U)$, one preceding $H(S(\tau_n))$ in time (because of the stable manifold) and one following (because of the unstable manifold, but the preimage of this point will have $\tau < \tau_{n-1}$)). This gives two sequences of points, one which converges to $H(p_s)$, with p_s a point on the stable manifold of q, and one which converges to $H(p_u)$, with p_u a point on the unstable manifold of q. Each point in each sequence is the image of a point of the orbit of S. Thus a point of the stable manifold of q and a point of the unstable manifold of q are in Λ. For a more detailed argument, see [14, pages 33 and 34]. The reasoning is taken from [2, Proof of *Theorem 4.3*] and the proof of a similar result in [16, Appendix]. From this in turn it follows that the entire stable manifold, its α-limit set, an orbit of the unstable manifold and its ω-limit set must all lie in Λ. The α-limit set of the stable manifold of q is the Kasner point which follows q in the BKL sequence. The stable manifold of q is the transition solution between them.

The other possibility is that Λ contains a flat Kasner point T_i. $\overline{I}_i$ is a center manifold of T_i in $\overline{U}$.

Proposition 2. The ω-limit set of any solution in I_i consists of the point T_i.

Proof of Proposition 2. It was already shown that the ω-limit set of any solution in I_i consists of a single Kasner point. The analysis of the orbit structure in the neighborhood of a non-flat Kasner point, q, showed that the stable manifold of q is a single orbit in one of the two-dimensional boundary sets, and that the ω-limit set of no other solution in $\overline{U}$ (besides q itself) is the single point q. The remaining possibilities are the flat Kasner points, T_i.

The function W^+ is strictly increasing on solutions in I_3. Therefore the value of W^+ on the ω-limit set of a solution in I_3 must be greater than the infimum of W^+ on I_3. So T_1 cannot be in the ω-limit set of a solution in I_3. The function L is strictly decreasing on solutions in I_3. Therefore the value of L on the ω-limit set of a solution in I_3 must be less than the supremum of L on I_3. So T_2 cannot be in the ω-limit set of a solution in I_3. But the ω-limit set of a solution in I_3 consists of a single Kasner point. The only remaining possibility is T_3, so T_3 must be the ω-limit point of all solutions in I_3.

Similar reasoning shows that the ω-limit point of all solutions in I_2 is T_2.

To find out what is the case for I_1, first recall the polarized case. This one-dimensional set is a single orbit whose α-limit set is Q_1 and whose ω-limit set is T_1. Now consider the (open and invariant) set of non-polarized solutions in I_1. The function Z (equation 5) is bounded from below ($Z > 0$) and it is strictly decreasing on solutions in this set. Therefore it must approach a finite value on the ω-limit set of one of these solutions. This is impossible at T_2 and at T_3. However, at T_1, if $(3\Sigma_-^2 + N_+^2) \to 0$ fast enough along solutions compared to $3N_-^2 - N_+^2$ then it is possible for Z to approach a finite value. Since the ω-limit set must consist of a single Kasner point, and all the other possibilities have been ruled out, then this must be the case. □

Proposition 3. The α-limit set of any solution in I_i consists of a point in $\mathcal{K}_i$.

Proof of Proposition 3. Given a point $q \in \mathcal{K}_i$, it has been shown that the α-limit sets of solutions on a two-dimensional submanifold of I_i consist of the point q. It has been ruled out that the α-limit set of any other solution in $\overline{U}$ can consist of the single point q. It has been shown that the α-limit set of a solution in I_i consists of a single Kasner point. The point T_i is ruled out by the existence of a monotonic function on solutions of I_i, since the ω-limit set of solutions of I_i consists of the point T_i. The remaining two flat points, T_j, $j \neq i$ can be ruled out in each case by considering $H_j(I_i \cap N_j)$, defined in the next paragraph. □

Now consider the case that a flat Kasner point T_i is in Λ. Let O_C be the intersection of $\overline{I}_i$ with a

neighborhood, $N_i \subset \overline{U}$, of T_i. Let x_1, x_2, x_3 be coordinates on O_C. Let

$$O_i = \{(x_1, x_2, x_3, y) : x \in O_C, 0 \leq y < \epsilon\}.$$

Let $x(\tau)$ satisfy the nonlinear system restricted to 0_C and $y(\tau) = y_0\, e^{\tau}$. Let ϵ and N_i be such that the nonlinear system on N_i is topologically equivalent to this system on O_i, with homeomorphism $H_i : N_i \to O_i$. If x is on the boundary of I_i then $H_i^{-1}(x, y)$ is on the boundary of U. If $x \in I_i$ and $0 < y < \epsilon$ then $H_i^{-1}(x, y) \in U$. Since there exists a (small enough) neighborhood of T_i such that no solution in I_i lies entirely within the neighborhood and since all solutions in I_i converge to T_i, I_i plays the role that the stable manifold played for the case that a non-flat Kasner point is in Λ. Then similar reasoning leads to the result that the unstable manifold of T_i and an orbit other than T_i in the closure of the center manifold along with the limit sets of these two solutions must be in Λ. Q_i is the Kasner point preceding T_i in the BKL sequence, and the unstable manifold of T_i is the transition solution between them. This ends the proof of the Theorem.

5 From oscillatory to mixmaster

It was found in the previous section that Λ contains the following: at least one non-flat Kasner point (q_k), the stable manifold of q_k, its α-limit set (q_{k+1}), an orbit in the unstable manifold of q_k and its ω-limit set (which is a single Kasner point). If q_{k+1} is a non-flat point, then its stable manifold is also in Λ, and so on. Let $q_k \to q_{k+1}$ stand for the union of q_k and its stable manifold and q_{k+1}. Then

$$q_k \to q_{k+1} \to q_{k+2} \to q_{k+3} \to q_{k+4} \cdots \tag{6}$$

is in Λ. This BKL sequence of Kasner points can end at a flat point, can be periodic, or can be non-ending and non-periodic [10, 17, 18].

For this class of solutions to fit the standard picture of mixmaster dynamics it must be the case that orbits in the unstable manifold of q_k that do not belong to $\mathcal{A}$ are not in Λ, for generic solutions in U. That is, there should be no points of vacuum Bianchi VI_0 and no points of magnetic Bianchi II in Λ. A polarized solution in U is exceptional. There is an orbit, Γ_1 (see the end of section 3), in its α-limit set that does not belong to $\mathcal{A}$. Γ_1 is the orbit of the vacuum polarized Bianchi VI_0 solution. It will now be shown that no other solutions in U have this type of exceptional behavior.

Proposition 4. No points in the orbit of any vacuum Bianchi VI_0 solution are in the α-limit set, Λ, of a non-polarized magnetic Bianchi VI_0 solution.

Proof of Proposition 4. The function Z (equation 5) is strictly decreasing on non-polarized solutions in U, so Z must be constant on Λ. Z is also strictly decreasing on non-polarized solutions in vacuum Bianchi VI_0. Since α-limit sets are the union of orbits then no point of a non-polarized vacuum Bianchi VI_0 orbit can be in Λ. The polarized vacuum Bianchi VI_0 orbit is also ruled out since the value of Z at points of this orbit equals the infimum of Z on the set of non-polarized solutions in U. $\square$

This means that if some $q_k \in \mathcal{K}_1$ is in Λ then the only orbits in its unstable manifold that can be in Λ are the two that are in vacuum Bianchi II. These are precisely the orbits of the transition solutions between q_{k-1} and q_k. (Given a non-flat Kasner point q, there are two Kasner points which can immediately precede q in the BKL sequence.)

Two possibilities come to mind for showing that no points of magnetic Bianchi II are in Λ, for generic solutions in U. One possibility is more careful analysis of the orbit structure in $\overline{U}$ in the neighborhood of a Kasner point. Another is the following. If two functions analogous to Z could be found, both of which were strictly monotonic on (perhaps generic) solutions in U, one also monotonic on (perhaps generic) solutions in magnetic Bianchi II^+ and the other monotonic on (perhaps generic) solutions in magnetic Bianchi II^-, then the same result just obtained for $q_k \in \mathcal{K}_1$ would be obtained

for $q_k \in \mathcal{K}_2$ and for $q_k \in \mathcal{K}_3$. Only two orbits in the unstable manifold of q_k would be left as possibilities for being in Λ. These are the orbits of the transition solutions of the BKL map between the two possible q_{k-1}'s and q_k. One or the other of these orbits would have to be in Λ. Therefore a sequence of the following form would have to be in the α-limit set of generic magnetic Bianchi VI_0 solutions.

$$\ldots q_{k-2} \rightarrow q_{k-1} \rightarrow q_k \rightarrow q_{k+1} \rightarrow q_{k+2} \cdots \tag{7}$$

This sequence is either periodic or infinite, since, while the domain of the BKL map is the set of all Kasner points except for the three flat Kasner points, the range is the set of all Kasner points. Therefore the sequence can end (a point is in the range that is not in the domain), but it cannot begin (every point that is in the domain is in the range).

Showing that no points of magnetic Bianchi II belong to Λ would, furthermore, imply that if the sequence 7 ends at a T_i, that is, if T_i is in Λ, then so is a non-flat Kasner point in any neighborhood of T_i. As of now, this result can only be obtained for T_1.

Proposition 5. If T_1 is in the α-limit set, Λ, of a non-polarized magnetic Bianchi VI_0 solution, then for any neighborhood, N_ϵ, of T_1, no matter how small, there is a non-flat Kasner point q such that $q \in N_\epsilon \cap \Lambda$.

Proof of Proposition 5. In the proof of the theorem, it was found that if $T_1 \in \Lambda$ then an orbit in $\overline{I}_1$ is also in Λ. But it has now been shown that no points of vacuum Bianchi VI_0 are in Λ. Therefore, an orbit in vacuum Bianchi II is in Λ. Furthermore, the reasoning in the proof of the theorem shows that for any neighborhood N_δ of T_1 that is small enough, there is a vacuum Bianchi II orbit in Λ which intersects the boundary of N_δ in $\overline{U}$. But then the limit sets of the orbit must also be in Λ. So for any neighborhood, N_ϵ, of T_1, choose N_δ with the following properties:

i. N_δ is a neighborhood of T_1 in $\overline{U}$.
ii. No solutions in I_1 are entirely contained in N_δ and the unstable manifold of T_1 is not entirely contained in N_δ.
iii. $N_\delta \subset N_1$, where N_1 is a neighborhood of T_1 which is topologically equivalent to O_1, as given at the end of section 4.
iv. The limit points of all vacuum Bianchi II orbits which intersect $\overline{N}_\delta$ are contained in N_ϵ.

None of these limit points is T_1. All of these limit points are non-flat Kasner points. At least one of these limit points is in Λ. □

6 Strong cosmic censorship

While the results of this paper do not give a complete characterization of the α-limit set of each magnetic Bianchi VI_0 solution, they are sufficient for showing that none of these solutions has an extension past the initial singularity. That this is the case follows from the unboundedness of the Kretschmann scalar, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, in a neighborhood of the singularity. That the Kretschmann scalar is unbounded in a neighborhood of the singularity follows from the unboundedness of the expansion scalar in a neighborhood of the singularity and the presence of a non-flat Kasner point in the α-limit set (so there is a sequence of points on the orbit of any magnetic Bianchi VI_0 which both "goes to the singularity" and converges to a non-flat Kasner point). The argument is the same as in the proof of *Theorem 5.1* in [2]. See also [2, equations (3.5) and (3.6)]. The ratio between the Kretschmann scalar and the fourth power of the expansion scalar can be expressed as a polynomial in Σ_+, Σ_-, N_+, N_- and H. This ratio is nonvanishing at a non-flat Kasner point, and the result follows.

The same argument shows that the Kretschmann scalar is unbounded in a neighborhood of the singularity in each of the solutions considered in this paper which has a non-flat Kasner point in its α-limit set, and so such a solution cannot be extended past the singularity. This is the case for each

vacuum Bianchi VI_0 solution and each of the magnetic Bianchi II solutions considered in this paper. It is also the case for "most" of the vacuum Bianchi II solutions, the magnetic Bianchi I solutions and the Kasners themselves.

The exceptions are the flat Kasner spacetimes, the rotationally symmetric vacuum Bianchi II spacetimes and the rotationally symmetric magnetic Bianchi I spacetimes. The α-limit set of each of these consists of a single flat Kasner point and the argument just given is inconclusive in these cases (the ratio between the Kretschmann scalar and the fourth power of the expansion scalar vanishes). It turns out that each of these spacetimes can, in fact, be extended past the initial singularity. It is already known that the flat Kasner spacetimes and the rotationally symmetric vacuum Bianchi II spacetimes [19] can be extended. An extension for the rotationally symmetric magnetic Bianchi I spacetimes is given in the appendix.

7 Conclusion

It has been shown in this paper that any spatially homogeneous solution to Einstein's equation of Bianchi type VI_0 with source consisting of a magnetic field orthogonal to the two commuting Killing vector fields has an oscillatory approach to the initial singularity and has no extension past the singularity. In the formulation which has been used, other spatially homogeneous solutions lie on the boundary of magnetic Bianchi VI_0. All of the boundary solutions are convergent rather than oscillatory, and most are inextendible past the initial singularity. The only ones which are extendible are the rotationally symmetric vacuum Bianchi II solutions, the rotationally symmetric magnetic Bianchi I solutions and the flat Kasners. Since these are non-generic in the class, strong cosmic censorship is supported.

The methods used here to characterize the approach to the singularity are those of dynamical systems analysis. One would like to obtain a rigorous proof that generic non-polarized magnetic Bianchi VI_0 solutions are mixmaster. This result could be obtained by finding two functions which are strictly monotonic on all or at least generic magnetic Bianchi VI_0 solutions as formulated here and also, in turn, on the two classes of magnetic Bianchi type II solutions which lie on the boundary. The existence of such functions would exclude these solutions of the boundary from the α-limit sets of magnetic Bianchi VI_0 solutions. Alternatively, one could try to show that this class is mixmaster by other methods. It appears that such a result has been recently obtained by Ringström for solutions of vacuum Bianchi types VIII and IX [13].

It is hoped that this work will contribute to the understanding of the nature of cosmological singularities in classical General Relativity. In particular, the validity in the generic case of the description of the approach to the cosmological singularity given by Belinskii, Khalatnikov and Lifshitz is still an open question. Whether or not the methods of dynamical systems analysis turn out to be directly useful in studying this question in spatially inhomogeneous classes of cosmological solutions to Einstein's equation, they have added to our understanding of the spatially homogeneous case, which gives a surer foundation for further study.

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Appendix

The metric of a rotationally symmetric Bianchi I spacetime with a magnetic field orthogonal to the plane of rotational symmetry can be written as follows [3, Appendix C, Case 6].

$$g = -A^2\,dt^2 + t^2 A^{-2}\,dx^2 + A^2\,dy^2 + A^2\,dz^2 \tag{8}$$

with $A = 1 + \frac{1}{4}B^2 t^2$. The Maxwell tensor is $F = B\,(dy \otimes dz - dz \otimes dy)$, with B a constant. In the direction of the singularity ($t \to 0$) the generalized Kasner exponents converge to $(1, 0, 0)$. This is the point T_1 in the formulation used in the present paper. In the opposite time direction ($t \to \infty$) the generalized Kasner exponents converge to $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. This is the point Q_1 in the formulation used in the present paper. These solutions were given by Rosen [20] and studied by Jacobs [21], but the author is not aware of a published extension of these spacetimes.

Here is a coordinate transformation that allows each of these spacetimes (inequivalent spacetimes result from different values of the constant $|B|$) to be extended.

$$\tilde{t} = t\,\cosh x \qquad\qquad \tilde{x} = t\sinh x \tag{9}$$

In the original spacetime, $t^2 = \tilde{t}^2 - \tilde{x}^2$. But if a function D is defined by $D = \tilde{t}^2 - \tilde{x}^2$, and if the function A is now defined $A = 1 + \frac{1}{4}B^2 D$, then the transformed metric

$$g = A^2\,(-d\tilde{t}^2 + d\tilde{x}^2) + (-B^2 + B^4 D\frac{32 - B^4 D^2}{256 A^2})(\tilde{x}\,d\tilde{t} - \tilde{t}\,d\tilde{x})^2 + A^2\,dy^2 + A^2\,dz^2 \tag{10}$$

is a solution to Einstein's equation in the region $A > 0$. This is an extension of the original solution. The region defined by $0 \geq D > -\frac{4}{B^2}$ is not in the original spacetime. The singularity at $A = 0$ is a curvature singularity. Both the Kretschmann scalar and $R_{\alpha\beta}R^{\alpha\beta}$ are unbounded as $A \to 0$.

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = (20 - 6B^2 D + \frac{3}{4}B^4 D^2)\frac{B^4}{A^8} \qquad R_{\alpha\beta}R^{\alpha\beta} = \frac{4B^4}{A^8} \tag{11}$$

$\partial_{\tilde{t}}$ is everywhere timelike, but $\partial_{\tilde{x}}$ is not everywhere spacelike. However, $g_{\tilde{t}\tilde{t}}\,g_{\tilde{x}\tilde{x}} - g_{\tilde{t}\tilde{x}}^2 = -1$. Thus the two-dimensional surface defined by constant y and z is everywhere timelike.

The singularity is timelike.

Along the $\tilde{x}$-axis the spacetime becomes singular a finite distance from the origin in either direction.

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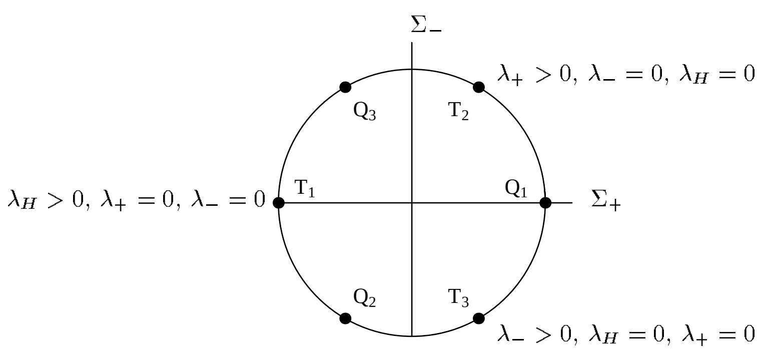

Figure 1: The Kasner circle. The arc between T_2 and T_3 containing Q_1 is $\mathcal{K}_1$. On $\mathcal{K}_1$, $\lambda_H < 0$, while $\lambda_+ > 0$ and $\lambda_- > 0$. The arc between T_3 and T_1 containing Q_2 is $\mathcal{K}_2$. On $\mathcal{K}_2$, $\lambda_+ < 0$, while $\lambda_- > 0$ and $\lambda_H > 0$. The arc between T_1 and T_2 containing Q_3 is $\mathcal{K}_3$. On $\mathcal{K}_3$, $\lambda_- < 0$, while $\lambda_H > 0$ and $\lambda_+ > 0$.